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Conformally flat, anisotropic spheres in general relativity

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Abstract. We examine anisotropic (principal stresses unequal), conformally flat, spherically symmetric interior solutions. The properties of several solutions are compared with those of the conformally flat Schwarzschild interior.

1. Introduction

Certainly no real astrophysical object is composed of a perfect fluid. Despite this, perfect fluid space–times have been widely studied as models, for instance, of neutron stars. In most astrophysical applications, it seems that perfect fluid models are adequate. However, possibly important changes in properties may occur when dealing with non-perfect fluid sources in highly compact bodies.

Recently, theoretical work on realistic stellar models has suggested that supradense stellar matter may be locally anisotropic (principal stresses unequal). This anisotropy could be the result of, for instance, a neutron crystalline core. Motivated by this possibility, several workers have examined the properties of locally anisotropic matter in strong gravitational fields. Bowers and Liang (1974), Herrera *et al* (1979) and Cosenza *et al* (1981) have examined how anisotropic matter affects the critical mass, maximum redshift, stability, etc of highly compact bodies. They have determined that in many cases the maximum equilibrium mass and surface redshift are increased over the isotropic (perfect fluid) values. Concerning stability, it was found that certain models were more stable if the matter was anisotropic whereas other models were less stable. Yodzis *et al* (1973) have found that naked singularities can occur in the spherical gravitational collapse of anisotropic matter.

Since realistic equations of state in supradense nuclear matter are not known even in the relatively simple perfect fluid case, the present author (Stewart 1981) has investigated a large collection of anisotropic interior solutions in an attempt to glean from them some generic behaviour. It would seem that, perhaps contrary to one's intuition, the tangential stresses are more important in support against gravitational collapse than the radial stress.

Given the difficulty in prescribing a realistic equation of state for anisotropic matter, the common procedure for finding such models has been to specify an *ad hoc* relation between the tangential and radial stresses, or to specify a relation involving the metric functions. Neither of these methods is completely satisfactory. In this paper we look for anisotropic models in which the mass distribution is specified and in which the mathematical requirement of conformal flatness is made. We do not assert that conformally flat solutions are any more physically reasonable than perfect fluid

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solutions. However, there has been much recent interest in conformally flat space–times (Banerjee and Santos 1981a, b, Reddy 1979). As is well known, the constant density Schwarzschild solution is the unique conformally flat, static perfect fluid space–time. However, if the perfect fluid requirement is relaxed, one may in principle obtain many conformally flat models. In order to ensure that the models are physically reasonable, the mass distributions that we will examine contain parameters which can be chosen so that the models approach the constant density solution. Therefore, in a certain range of parameters, the solutions obtained approximate perfect fluid solutions.

It has been shown by Bondi (1964) that for perfect fluids† the largest value of the ratio of mass to radius is $\frac{4}{9}$. This maximum value is attained in the constant density solution. In this work we will examine how the maximum value of this ratio changes for other conformally flat, but non-perfect fluid, solutions.

This paper is organised as follows. In § 2 we examine the Einstein field equations in the case of a static, spherically symmetric energy–momentum tensor and a conformally flat line element. We utilise the remaining freedom in the field equations to choose the functional form of the mass distribution. Section 4 contains four example solutions and an examination of some of their properties. Next we look at conformally flat, slowly rotating solutions. The last section contains a short discussion of the results.

2. Field equations and boundary conditions

We begin by examining the most general static, spherically symmetric, conformally flat line element as given by Banerjee and Santos (1981a):

$$ds^2 = e^{2\nu}[(c\rho^2+b)^2\, dt^2 - d\rho^2 - \rho^2\, d\Omega^2], \tag{1a}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{1b}$$

where

$$\nu = \nu(\rho), \qquad b, c \text{ are constants.} \tag{1c}$$

We will perform a coordinate transformation to Schwarzschild curvature coordinates

$$e^{\nu}\rho = r. \tag{2}$$

Then

$$ds^2 = e^{2\varphi}(c\, e^{-2\varphi}r^2 + b)^2\, dt^2 - (1 - r\varphi')^2\, dr^2 - r^2\, d\Omega^2 \tag{3a}$$

where

$$\varphi(r) = \nu(\rho), \tag{3b}$$

$$' \equiv d/dr. \tag{3c}$$

In order to check if this line element is indeed conformally flat for any $\varphi(r)$, we use the fact that for the line element

$$ds^2 = e^{\mu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\Omega^2, \tag{4a}$$

† This work applies only to perfect fluid solutions in which the density is a non-increasing function of the radial coordinate.

where

$$\mu = \mu(r), \qquad \lambda = \lambda(r), \tag{4b, c}$$

the space–time is conformally flat if (Takeno 1966)

$$\frac{\exp\lambda}{r^2} - \frac{1}{r^2} - \frac{\mu'^2}{4} + \frac{\mu'\lambda'}{4} - \frac{\mu''}{2} - \frac{(\lambda' - \mu')}{2r} = 0. \tag{5}$$

Direct substitution of equation (3) into equation (5) verifies the conformal flatness of (3).

If we define

$$\varphi' \equiv r^{-1} - (r^2 - 2mr)^{-1/2}, \tag{6a}$$

$$m = m(r) \equiv \text{mass distribution}, \tag{6b}$$

then equation (3) can be rewritten as

$$ds^2 = e^{2\varphi}(c\, e^{-2\varphi} r^2 + b)^2\, dt^2 - (1 - 2m/r)^{-1}\, dr^2 - r^2\, d\Omega^2. \tag{7}$$

Next we adopt the form of the energy–momentum tensor for an anisotropic fluid used by Letelier (1980):

$$T_{\mu\nu} = \mu u_\mu u_\nu + (\sigma - \pi) x_\mu x_\nu - \pi(g_{\mu\nu} - u_\mu u_\nu) \tag{8}$$

where u_μ is the anisotropic fluid four-velocity, x_μ is a space-like unit four-vector that points in the direction of the anisotropy, μ is the usual rest energy density of the fluid, π is the pressure on a plane perpendicular to the anisotropy direction, and σ is the pressure along the anisotropy direction.

In the static case, the field equations for the line element (7) are (Tolman 1962) (units $8\pi G = c = 1$)

$$T_0^0 = \mu = 2m'/r^2, \tag{9a}$$

$$-T_1^1 = \sigma = (cr^2 + b\, e^{2\varphi})^{-1}\left\{4c\left(1 - \frac{2m}{r}\right) - \frac{2}{r^2}(cr^2 - b\, e^{2\varphi})\left[1 - \frac{2m}{r} - \left(1 - \frac{2m}{r}\right)^{1/2}\right]\right\} - \frac{2m}{r^3}, \tag{9b}$$

$$-T_2^2 = -T_3^3 = \pi, \tag{9c}$$

$$\pi - \sigma \equiv \Delta = -r[2m/r^3]'. \tag{9d}$$

Examination of the system (9) reveals that we have four unknowns (μ, σ, π, m) and only three independent equations (9a, b, c). Therefore the freedom exists to choose, for example, the functional form of the mass distribution, $m(r)$.

The boundary conditions, in order to match the interior solutions to the vacuum Schwarzschild solution, are (Bonnor and Vickers 1981):

$$g_{00}, g_{22}, g_{33} \text{ are } C', \qquad g_{11} \text{ is } C^0,$$

on the boundary of the body, $r = a$.

Application of these conditions yields

$$c = \frac{1}{2}\frac{e^{\varphi(a)}}{a^2}\left[\frac{3M}{a} - 1 + \left(1 - \frac{2M}{a}\right)^{1/2}\right], \qquad b = \frac{1}{2} e^{-\varphi(a)}\left[1 - \frac{3M}{a} + \left(1 - \frac{2M}{a}\right)^{1/2}\right]. \tag{10a, b}$$

3. The mass distribution

We proceed by choosing some physically reasonable forms for the mass distribution, $m(r)$. φ is then found by the integration of equation (6a). The criteria to be used here to determine physical reasonableness are the following.

(i) $m(r) \geqslant 0$. Only positive mass is allowed.

(ii) $m''r - 2m' \leqslant 0$. This is equivalent to requiring non-increasing (as r increases) mass density, i.e. $\mu' \leqslant 0$.

(iii) $\lim_{r\to 0} m(r)/r = 0$. Only finite mass density is allowed.

(iv) All physical variables are finite everywhere in the body.

(v) We require that g_{00}, g_{11} are finite and non-zero everywhere in the interior, with no changes of sign nor loss of reality allowed.

(vi) Finally we require that the pressure is finite and non-negative.

Integration of the relation in requirement (ii) yields

$$m'(r) \leqslant Ar^2, \qquad A = \text{constant}.$$

Further integration, with requirement (iii) in mind, leaves

$$m(r) \leqslant Ar^3/3.$$

The equality holds for the constant density solution. One can see that (a) as long as $\lim_{r\to 0} 2m/r \to 0$ at least as fast as r^2; (b) e^{φ} is finite and non-zero; and (c) the constant b is non-zero, then $\sigma(r=0)$ is finite. In fact, assuming that

$$2m/r \approx \Gamma r^2 + O(r^3) \quad \text{as } r \to 0, \qquad \Gamma = \text{constant}, \tag{11}$$

then

$$\sigma(r \to 0) = (4c/b)\, e^{-2\varphi(r=0)} - 2\Gamma. \tag{12}$$

It is clear that as long as Γ, C, and $e^{-2\varphi(r=0)}$ are finite and non-zero, the radial stress is infinite if $2M/a = \frac{8}{9}$, where b vanishes.

We next present four examples of mass distributions which satisfy the above requirements.

Example (1)

$$2m(r) = r[1 - (\sin^2 Kr)/K^2r^2], \qquad K = \text{constant}. \tag{13}$$

Example (2)

$$2m(r) = r \tanh^2 Kr, \qquad K = \text{constant}, \tag{14}$$

Example (3)

$$2m(r) = r[1 - \exp(-Br^2)], \qquad B = \text{constant}, \tag{15}$$

Example (4)

$$2m(r) = A(\tfrac{1}{3}r^3 - \alpha r^5/5a^2), \qquad A, \alpha = \text{constant}. \tag{16}$$

We will examine some properties of the solutions found using the above forms for the mass distribution in the next section.

4. The solutions

In this section, we will examine the properties of the solutions, with a bit more emphasis on the two-parameter solution (4).

Example (1)

In this example we choose as the mass distribution the function

$$2m(r) = r[1 - (\sin^2 Kr)/K^2r^2]. \tag{17}$$

Thus from (9*a*)

$$\mu(r) = \frac{1}{r^2}\left(1 - \frac{\sin 2Kr}{Kr} - \frac{\sin^2 Kr}{K^2r^2}\right). \tag{18}$$

We can integrate equation (6*a*) to obtain

$$e^{2\varphi} = \tfrac{1}{4}K^2r^2 \cot^2 \tfrac{1}{2}Kr \tag{19}$$

where the constant of integration has been chosen so that

$$e^{2\varphi}|_{r=0} = 1.$$

Then

$$\Delta = \frac{2}{r^2}\left(1 - 2\frac{\sin^2 Kr}{K^2r^2} + \frac{1}{2}\frac{\sin 2Kr}{Kr}\right). \tag{20}$$

The constant K is found from the boundary conditions by solving

$$(\sin Ka)/Ka = (1 - 2M/a)^{1/2}. \tag{21}$$

Since for $2M/a = \frac{8}{9}$ the central stresses are infinite, we will require that $2M/a < \frac{8}{9}$. We can solve (21) graphically to find the restriction of K to be approximately

$$K \leqslant 29\pi/40a. \tag{22}$$

The mass density and stresses are non-negative, monotonically decreasing functions of the radial coordinate, at least for a range of K. Δ, the deviation from perfect fluidity, is a non-negative function of r. $\Delta(r = 0)$ vanishes. For different values of K, Δ can be both increasing and decreasing in the interior. In the limit $K \to 0$ flat space–time results. For K small but non-zero, the solution differs only slightly from the constant density solution.

Example (2)

The mass distribution was taken to be

$$2m(r) = r \tanh^2 Kr. \tag{23}$$

From (9*a*)

$$\mu(r) = r^{-2} \tanh^2 Kr + r^{-1} 2K \tanh Kr \operatorname{sech}^2 Kr. \tag{24}$$

Integration of equation (6*a*) yields

$$e^{2\varphi} = \exp\left(-2 \sum_{n=1}^{\infty} \frac{(Kr)^{2n}}{2n(2n)!}\right) \tag{25}$$

where we have again chosen the constant of integration so that

$$e^{2\varphi}|_{r=0} = 1.$$

From (9*d*)

$$\Delta = \frac{1}{r^2}\frac{\tanh Kr}{\cosh^2 Kr}(\sinh 2Kr - 2Kr). \tag{26}$$

Application of the boundary conditions yields

$$K = \frac{1}{a}\tanh^{-1}\left(\frac{2M}{a}\right)^{1/2} = \frac{1}{2a}\ln\left(\frac{1+(2M/a)^{1/2}}{1-(2M/a)^{1/2}}\right). \tag{27}$$

The restriction that $2M/a < \frac{8}{9}$ implies for K

$$Ka \leqslant 1.76. \tag{28}$$

The mass density and stresses are non-negative, monotonically decreasing functions of r at least for a range of K. Δ is a non-negative function of r. $\Delta(r=0)$ vanishes. For different values of K, Δ can be both increasing and decreasing in the interior. In the limit $K \to 0$, again flat space–time results. For K small but non-zero, the solution differs only slightly from the constant density solution.

Example (3)

In this case the mass distribution is given by

$$2m(r) = r[1-\exp(-Br^2)]. \tag{29}$$

Thus

$$\mu(r) = r^{-2}[1-\exp(-Br^2)] + 2B\exp(-Br^2). \tag{30}$$

Integration of (6*a*) yields

$$e^{2\varphi} = \exp\left(-2\sum_{n=1}^{\infty}\left(\frac{Br^2}{2}\right)\frac{1}{nn!}\right). \tag{31}$$

Again

$$e^{2\varphi}|_{r=0} = 1.$$

From (9*d*)

$$\Delta = (2/r^2)[1-\exp(-Br^2) - Br^2\exp(-Br^2)]. \tag{32}$$

The boundary conditions yield

$$B = -a^{-2}\ln(1-2M/a). \tag{33}$$

The condition that $2M/a < \frac{8}{9}$ implies

$$B < a^{-2}\ln 9. \tag{34}$$

The mass density and stresses are non-negative, monotonically decreasing functions of r, at least for a range of B. Δ is a non-negative function of r; $\Delta(r=0)$ vanishes. For different values of B, Δ can be both increasing and decreasing in the interior. In the limit $B \to 0$, flat space–time results. For B small but non-zero the solution differs only slightly from the constant density solution.

Example (4)

The previous three solutions each contained one parameter which was related, via the boundary conditions, to the value of $2M/a$. In this example, however, we present in more detail a two-parameter solution. The mass distribution is

$$2m(r) = A(\tfrac{1}{3}r^3 - \alpha r^5/5a^2), \qquad A, \alpha = \text{constant}. \tag{35}$$

Then

$$\mu(r) = A(1 - \alpha r^2/a^2), \qquad 0 \leqslant \alpha \leqslant 1. \tag{36}$$

The restrictions on α are due to the requirement that $\mu(r) \geqslant 0$ for $0 \leqslant r \leqslant a$. Also

$$\Delta = \tfrac{2}{5}A\alpha r^2 \to 0 \qquad \text{as } r \to 0. \tag{37}$$

Integrating (6a),

$$e^{2\varphi} = \tfrac{1}{2}[1 + (1 - \tfrac{1}{3}Ar^2 + \alpha Ar/5a^2)^{1/2} - \tfrac{1}{6}Ar^2], \qquad e^{2\varphi}|_{r=0} = 1. \tag{38}$$

The boundary conditions yield

$$A = (6M/a^3)(1 - \tfrac{3}{5}\alpha)^{-1}. \tag{39}$$

The constant b in equation (10b) is real as long as

$$2M/a \geqslant (1 - \tfrac{3}{5}\alpha)\tfrac{12}{5}\alpha. \tag{40}$$

A curious upper limit is placed upon the value of α by the requirement of non-negative stresses. For small values of $2M/a$, it is

$$10M/8a > \alpha. \tag{41}$$

Of course α can always be chosen to satisfy equation (41).

For $\alpha \to 0$, we have the incompressible fluid Schwarzschild solution. The stresses and mass density, for a range of $\alpha > 0$, are non-negative, monotonically decreasing functions of r. Δ is a non-negative, monotonically increasing function of r. $\Delta(r = 0)$ vanishes.

This model is almost certainly more stable than the Schwarzschild interior since the mass density for $\alpha > 0$ is a decreasing function. In table 1, we have listed the comparative values of the central stresses for selected values of the ratio $2M/a$. It can be readily seen that the central pressure in the Schwarzschild case is larger for a given value of $2M/a$ than in the $\alpha > 0$ model. This is another indication of increased stability.

Table 1. Comparison of values of $a^2\sigma$ $(r = 0)$ for the solution, (4), $(\alpha = 0.1)$ and the Schwarzschild solution $(\alpha = 0)$.

$2m/a$	$a^2\sigma$ (Schwarzschild)	$a^2\sigma$ (solution (4))
0.88	43.98	42.45
0.77	2.74	2.55
0.66	1.10	0.98
0.55	0.54	0.45
0.44	0.27	0.20
0.33	0.12	0.08
0.22	0.05	0.02

5. Slowly rotating conformally flat interior solutions

Much interest has been present recently in examining the properties of relativistic rotating perfect fluids. Due to the difficulties encountered by workers in obtaining a perfect fluid solution which would serve as a source for the Kerr metric, interest has developed in the examination of solutions of the Brill–Cohen (1966) 'slowly rotating' formalism. We have attempted to find solutions to the 'slow rotation' structure equations which were conformally flat and physically reasonable. Unfortunately, if one restricts one's attention to those interior solutions in which no singularities are present, it is found that no such slowly rotating conformally flat solutions exist. This is demonstrated in the appendix. Since slow rotation is an analytic limit of the general problem, it would seem that no singularity-free, conformally flat, rotating interior solution may be possible.

6. Conclusions

In this work we have examined static, spherically symmetric, conformally flat interior solutions. The form of the energy–momentum tensor that we have assumed is the most general allowed under the symmetries imposed upon the metric (static line element, spherical symmetry). Even after the imposition of the mathematical requirement of conformal flatness, freedom exists to specify the functional form of the mass distribution. We have used this freedom to develop four physically reasonable models of anisotropic fluids.

Analysis would seem to indicate that although the maximum value of the ratio $2M/a$ is not increased from its maximum perfect fluid value of $\frac{8}{9}$, the anisotropic fluid models are more stable than the constant density solution. In fact, the maximum ratio of $2M/a$ is independent of the particular choice for the (finite) mass distribution, since it follows almost solely from the boundary conditions on the constants c and b from equation (10). Thus one would expect this to be a property common to all physically reasonable conformally flat space–times.

Another rather interesting property of these solutions involves the Weyl tensor discontinuity across the boundary of the body. Shepley (1968) has examined the discontinuity of the Weyl tensor:

$$[C^{\alpha\beta}{}_{\gamma\delta}] \equiv C^{\mathrm{I}\alpha\beta}{}_{\gamma\delta} - C^{\mathrm{E}\alpha\beta}{}_{\gamma\delta}$$

where $C^{\mathrm{I}\alpha\beta}{}_{\gamma\delta}$ is the Weyl tensor of the interior space–time and $C^{\mathrm{E}\alpha\beta}{}_{\gamma\delta}$ is the Weyl tensor of the exterior (vacuum) space–time. He was able to show that for perfect fluid solutions the discontinuity was of type D across a fluid–vacuum boundary. In the case under consideration in this work this is also the case since $C^{\mathrm{I}\alpha\beta}{}_{\gamma\delta} = 0$ and $C^{\mathrm{E}\alpha\beta}{}_{\gamma\delta}$ is type D for the vacuum Schwarzschild solution.

Appendix

The conditions of slow rotation lead to the metric

$$ds^2 = -A\, dt^2 + B\, dr^2 + r^2[d\theta^2 + \sin^2\theta(d\phi - \Omega\, dt)^2]$$

where A, B and Ω are functions of r only. The conditions of conformal flatness are that the components of the Weyl tensor

$$C_{abcd} \equiv R_{abcd} + \tfrac{1}{6}R(g_{ac}g_{bd} - g_{ad}g_{bc}) - \tfrac{1}{2}(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc})$$

vanish where R_{ab} is the Ricci tensor ($R_{ab} \equiv R^{c}_{\ acb}$), R is the Ricci scalar ($R \equiv R^{a}_{\ a}$) and R_{abcd} is the Riemann tensor (a, b, c, d take values from 0 to 3). In particular

$$C_{1230} = R_{1230} + \tfrac{1}{6}R(g_{13}g_{20} - g_{10}g_{23}) - \tfrac{1}{2}(g_{13}R_{20} - g_{23}R_{10} + g_{20}R_{13} - g_{10}R_{23})$$

where $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

The only non-vanishing off-diagonal component of the metric tensor is

$$g_{03} = g_{30} = -2r^2 \sin^2 \theta\Omega; \qquad \text{thus } C_{1230} = R_{1230}.$$

This particular component of the Riemann tensor is given by

$$R_{1230} = -r^2 \cos\theta \sin\theta A\Omega'/(5r^2\Omega^2 \sin^2\theta + A).$$

Thus the only way that C_{1230} can vanish everywhere in the interior is if Ω = constant. If Ω is a non-vanishing constant, the interior will possess a ring-like singularity in the equatorial plane. Thus a singularity-free conformally flat, slowly rotating interior solution would seem to be impossible.

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